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An integrable symmetric (2+1)-dimensional Lotka–Volterra equation and a family of its solutions

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Abstract

A symmetric (2+1)-dimensional Lotka–Volterra equation is proposed. By means of a dependent variable transformation, the equation is firstly transformed into multilinear form and further decoupled into bilinear form by introducing auxiliary independent variables. A bilinear Bäcklund transformation is found and then the corresponding Lax pair is derived. Explicit solutions expressed in terms of pfaffian solutions of the bilinear form of the symmetric (2+1)-dimensional Lotka–Volterra equation are given. As a special case of the pfaffian solutions, we obtain soliton solutions and dromions.

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1. Introduction

There has been a good deal of interest in recent years in integrable systems in (2+1)-dimensions. Such systems have dimensional reductions to known integrable equations in the (1+1)-dimension. If the two spatial variables appear on an equal footing and hence allow such reductions in either variable one calls the (2+1)-dimensional system a strong generalization of the (1+1)-dimensional system. The most well-known example is the Davey–Stewartson (DS) equations [1, 2] which strongly generalize the nonlinear Schrödinger equation. Another interesting example is the Loewner–Konopelchenko–Rogers (LKR) equations [3] which strongly generalize the sine-Gordon equation. For the Korteweg–de Vries (KdV) equation, the most famous (2+1)-dimensional generalization, the Kadomtsev–Petviashvili (KP) equation [4], is only a weak generalizaton. Physically the KP equation arises in situations where one-dimensional motion governed by the KdV equation is weakly perturbed in a perpendicular direction. A second, less well studied, generalization of the KdV equation is the Nizhnik–Novikov–Veselov (NNV) equation [5, 6]

$$u_t + u_{xxx} + u_{yyy} + 3(\phi_{xx}u)_x + 3(\phi_{yy}u)_y = 0, \qquad u = \phi_{xy}. \tag{1}$$

On the line $y = x$, (1) reduces to the KdV equation. In this way one sees that (1) is a strong generalization of the KdV equation.

The property of strong two-dimensionality seems to be closely related to the existence of localized, exponentially decaying solutions (dromions). Indeed, this is the key feature which leads to the existence of the underlying plane-wave structure of such solutions in the DS equations [7–9] and in the NVN equation [10–12]. To our knowledge it seems that examples of strong (2+1)-dimensional generalizations of integrable equations shown in the literature are restricted to the continuous case. Therefore, it is quite natural to consider strong (2+1)-dimensional generalizations of discrete integrable equations.

It is well known that the Lotka–Volterra (LV) equation

$$u_t(n) + \mathrm{e}^{u(n)+u(n+1)} - \mathrm{e}^{u(n)+u(n-1)} = 0 \tag{2}$$

is one of the most important integrable lattices. In the literature, several (2+1)-dimensional (two continuous and one discrete) generalizations of equation (2) are available (see, e.g. [13–15]). In addition, there are two (2+1)-dimensional (two discrete and one continuous) generalization of the Lotka–Volterra lattice [14, 16]. However, all of these known (2+1)-dimensional generalizations of the LV equation are not strong generalizations.

The purpose of this paper is firstly to propose a symmetric (2+1)-dimensional LV equation, secondly, to present a Bäcklund transformation and Lax pair for it and then to derive solutions expressed in terms of pfaffians. As a special reduction of these solutions we obtain soliton and dromion solutions.

We now propose the following symmetric (2+1)-dimensional LV equation:

$$\begin{aligned}2u_t(m,n) &+ \mathrm{e}^{u(m,n)+\Delta_m^2\phi(m,n+1)} - \mathrm{e}^{-u(m,n)+\Delta_n^2\phi(m,n)} + \mathrm{e}^{u(m,n)+\Delta_n^2\phi(m+1,n)} - \mathrm{e}^{-u(m,n)+\Delta_m^2\phi(m,n)}\\ &+ \mathrm{e}^{-u(m,n)+\Delta_n^2\phi(m+1,n-1)} - \mathrm{e}^{u(m,n)+\Delta_m^2\phi(m-1,n)} + \mathrm{e}^{-u(m,n)+\Delta_m^2\phi(m-1,n+1)}\\ &- \mathrm{e}^{u(m,n)+\Delta_n^2\phi(m,n-1)} = 0, \qquad u(m,n) = \Delta_m\Delta_n\phi(m,n)\end{aligned} \tag{3}$$

where Δ_m and Δ_n are difference operators defined by

$$\Delta_m u(m,n) = u(m+1,n) - u(m,n), \qquad \Delta_n u(m,n) = u(m,n+1) - u(m,n).$$

Obviously, in the reduction $m = n$, (3) becomes (2). In the next sections, we shall present a Bäcklund transformation and Lax pair for (3) and then establish a broad class of solutions expressed in terms of pfaffians. Finally, by considering special cases of this class, we obtain explicit expressions for the soliton and dromion solutions.

2. Bilinear Bäcklund transformation and Lax pair

Using the dependent variable transformation

$$u(m,n) = \ln\frac{f(m+1,n+1)f(m,n)}{f(m+1,n)f(m,n+1)},$$

equation (3) is transformed into the multilinear form

$$\begin{aligned}&\sinh\left(\tfrac{1}{2}D_n\right)\left[\left(D_t\,\mathrm{e}^{\frac{1}{2}D_m} - \mathrm{e}^{D_n-\frac{1}{2}D_m} + \mathrm{e}^{D_n+\frac{1}{2}D_m}\right)f\cdot f\right]\cdot\left(\mathrm{e}^{\frac{1}{2}D_m}f\cdot f\right)\\ &\quad+\sinh\left(\tfrac{1}{2}D_m\right)\left[\left(D_t\,\mathrm{e}^{\frac{1}{2}D_n} - \mathrm{e}^{D_m-\frac{1}{2}D_n} + \mathrm{e}^{D_m+\frac{1}{2}D_n}\right)f\cdot f\right]\cdot\left(\mathrm{e}^{\frac{1}{2}D_n}f\cdot f\right) = 0,\end{aligned} \tag{4}$$

where the bilinear operators D_t^k and $\exp(D_n)$ are defined by [17–19]

$$D_t a\cdot b \equiv \left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)a(t)b(t')|_{t'=t}, \qquad \exp(\delta D_n)a(n)\cdot b(n)=a(n+\delta)b(n-\delta).$$

It is noted that for continuous systems, multilinear forms of type (4) have been proposed for a model equation for shallow water waves [20] and the fifth-order KdV equation [21]. Besides, a trilinear form for the NVN equation was given in [22].

Concerning (4), we have the following result:

Proposition 1. *The multilinear equation (4) has the Bäcklund transformation*

$$\mathrm{e}^{\frac12 D_m+\frac12 D_n} f\cdot g=\left(\lambda\,\mathrm{e}^{\frac12 D_m-\frac12 D_n}+\mu\,\mathrm{e}^{-\frac12 D_m-\frac12 D_n}-\frac{\mu}{\lambda}\,\mathrm{e}^{-\frac12 D_m+\frac12 D_n}\right)f\cdot g, \tag{5}$$

$$\left(2D_t-\lambda\,\mathrm{e}^{-D_n}+\frac{1}{\lambda}\,\mathrm{e}^{D_n}+\frac{\mu}{\lambda}\,\mathrm{e}^{-D_m}-\frac{\lambda}{\mu}\,\mathrm{e}^{D_m}+\gamma\right)f\cdot g=0, \tag{6}$$

where λ, μ and γ are arbitrary constants.

Proof. Let $f(m,n)$ be a solution of equation (4). If we can show that $g(m,n)$ given by equations (5) and (6) satisfies the relation

$$\begin{aligned}
P \equiv{}& \left\{\sinh\left(\tfrac12 D_n\right)\left[\left(D_t\,\mathrm{e}^{\frac12 D_m}-\mathrm{e}^{D_n-\frac12 D_m}+\mathrm{e}^{D_n+\frac12 D_m}\right)f\cdot f\right]\cdot\left(\mathrm{e}^{\frac12 D_m}f\cdot f\right)\right.\\
&\left.+\sinh\left(\tfrac12 D_m\right)\left[\left(D_t\,\mathrm{e}^{\frac12 D_n}-\mathrm{e}^{D_m-\frac12 D_n}+\mathrm{e}^{D_m+\frac12 D_n}\right)f\cdot f\right]\cdot\left(\mathrm{e}^{\frac12 D_n}f\cdot f\right)\right\}\\
&\times\left[\mathrm{e}^{\frac12 D_n}\left(\mathrm{e}^{\frac12 D_m}g\cdot g\right)\cdot\left(\mathrm{e}^{\frac12 D_m}g\cdot g\right)\right]-\left\{\sinh\left(\tfrac12 D_n\right)\left[\left(D_t\,\mathrm{e}^{\frac12 D_m}-\mathrm{e}^{D_n-\frac12 D_m}\right.\right.\right.\\
&\left.\left.+\mathrm{e}^{D_n+\frac12 D_m}\right)g\cdot g\right]\cdot\left(\mathrm{e}^{\frac12 D_m}g\cdot g\right)+\sinh\left(\tfrac12 D_m\right)\left[\left(D_t\,\mathrm{e}^{\frac12 D_n}-\mathrm{e}^{D_m-\frac12 D_n}\right.\right.\\
&\left.\left.\left.+\mathrm{e}^{D_m+\frac12 D_n}\right)g\cdot g\right]\cdot\left(\mathrm{e}^{\frac12 D_n}g\cdot g\right)\right\}\left[\mathrm{e}^{\frac12 D_n}\left(\mathrm{e}^{\frac12 D_m}f\cdot f\right)\cdot\left(\mathrm{e}^{\frac12 D_m}f\cdot f\right)\right]=0,
\end{aligned}$$

then equations (5) and (6) form a BT. In fact, by using bilinear operator identities (A.1)–(A.10) and (5), (6), we can show that

$$\begin{aligned}
P ={}& \sinh\left(\frac12 D_n\right)\left\{\left[\left(2D_t\mathrm{e}^{\frac12 D_m}-\mathrm{e}^{D_n-\frac12 D_m}+\mathrm{e}^{D_n+\frac12 D_m}\right)f\cdot f\right]\left[\mathrm{e}^{\frac12 D_m}g\cdot g\right]\right.\\
&\left.-\left[\mathrm{e}^{\frac12 D_m}f\cdot f\right]\left[\left(2D_t\,\mathrm{e}^{\frac12 D_m}-\mathrm{e}^{D_n-\frac12 D_m}+\mathrm{e}^{D_n+\frac12 D_m}\right)g\cdot g\right]\right\}\cdot\left[\mathrm{e}^{\frac12 D_m}f\cdot f\right]\left[\mathrm{e}^{\frac12 D_m}g\cdot g\right]\\
&+\sinh\left(\frac12 D_m\right)\left\{\left[\left(-\mathrm{e}^{D_m-\frac12 D_n}+\mathrm{e}^{D_m+\frac12 D_n}\right)f\cdot f\right]\left[\mathrm{e}^{\frac12 D_n}g\cdot g\right]\right.\\
&\left.-\left[\mathrm{e}^{\frac12 D_n}f\cdot f\right]\left[\left(-\mathrm{e}^{D_m-\frac12 D_n}+\mathrm{e}^{D_m+\frac12 D_n}\right)g\cdot g\right]\right\}\cdot\left[\mathrm{e}^{\frac12 D_n}f\cdot f\right]\left[\mathrm{e}^{\frac12 D_n}g\cdot g\right]\\
={}& \sinh\left(\frac12 D_n\right)\left\{2\sinh\left(\frac12 D_m\right)\left[\left(2D_t-\lambda\,\mathrm{e}^{-D_n}+\frac1\lambda\,\mathrm{e}^{D_n}\right)f\cdot g\right]\cdot fg\right\}\cdot\left[\mathrm{e}^{\frac12 D_m}fg\cdot fg\right]\\
&+\sinh\left(\frac12 D_m\right)\left\{2\sinh\left(\frac12 D_n\right)\left[\left(\frac{\mu}{\lambda}\,\mathrm{e}^{-D_m}-\frac{\lambda}{\mu}\,\mathrm{e}^{D_m}\right)f\cdot g\right]\cdot fg\right\}\cdot\left[\mathrm{e}^{\frac12 D_n}fg\cdot fg\right]\\
={}& \sinh\left(\frac12 D_n\right)\left\{2\sinh\left(\frac12 D_m\right)\left[\left(2D_t-\lambda\,\mathrm{e}^{-D_n}+\frac1\lambda\,\mathrm{e}^{D_n}+\frac{\mu}{\lambda}\,\mathrm{e}^{-D_m}-\frac{\lambda}{\mu}\,\mathrm{e}^{D_m}\right)f\cdot g\right]\cdot fg\right\}\\
&\cdot\left[\mathrm{e}^{\frac12 D_m}fg\cdot fg\right]\\
={}& 0.
\end{aligned}$$

Thus we have completed the proof of proposition 1. □

Starting from the bilinear BT (5)–(6), we can derive a Lax pair for the symmetric (2+1)-dimensional LV equation (3). First, we define

$$\psi(m,n)=f(m,n)/g(m,n) \qquad \text{and} \qquad u(m,n)=\ln\frac{g(m+1,n+1)g(m,n)}{g(m+1,n)g(m,n+1)}$$

and rewrite (5), (6) in terms of these variables. After some calculation we obtain the following Lax pair for (3):

$$\psi(m+1,n+1)=\lambda\psi(m+1,n)\,\mathrm{e}^{-u(m,n)}+\mu\psi(m,n)-\frac{\mu}{\lambda}\psi(m,n+1)\,\mathrm{e}^{-u(m,n)}, \tag{7}$$

$$\begin{aligned}2\psi_t(m,n)&-\lambda\psi(m,n-1)\,\mathrm{e}^{\Delta_n(\phi(m,n)-\phi(m,n-1))}+\frac{1}{\lambda}\psi(m,n+1)\,\mathrm{e}^{\Delta_n(\phi(m,n)-\phi(m,n-1))}\\&+\frac{\mu}{\lambda}\psi(m-1,n)\,\mathrm{e}^{\Delta_m(\phi(m,n)-\phi(m-1,n))}-\frac{\lambda}{\mu}\psi(m+1,n)\,\mathrm{e}^{\Delta_m(\phi(m,n)-\phi(m-1,n))}\\&+\gamma\psi(m,n)=0.\end{aligned} \tag{8}$$

We have also checked that the compatibility condition of (7) and (8) yields the symmetric (2+1)-dimensional LV equation (3). Without loss of generality, we may choose $\lambda=1,\mu=-1$ and $\gamma=0$. In this case, Lax pair (7) and (8) is reduced to

$$\psi(m+1,n+1)+\psi(m,n)=\mathrm{e}^{-u(m,n)}[\psi(m+1,n)+\psi(m,n+1)], \tag{9}$$

$$\begin{aligned}2\psi_t(m,n)=\;&\psi(m,n-1)\,\mathrm{e}^{\Delta_n(\phi(m,n)-\phi(m,n-1))}-\psi(m,n+1)\,\mathrm{e}^{\Delta_n(\phi(m,n)-\phi(m,n-1))}\\&+\psi(m-1,n)\,\mathrm{e}^{\Delta_m(\phi(m,n)-\phi(m-1,n))}-\psi(m+1,n)\,\mathrm{e}^{\Delta_m(\phi(m,n)-\phi(m-1,n))}.\end{aligned} \tag{10}$$

In fact, the variable transformation

$$\psi(m,n)\longrightarrow\lambda^{n-m}(-\mu)^m\,\mathrm{e}^{-\frac{1}{2}\gamma t}\psi(m,n)$$

transforms (7)–(8) into (9)–(10). In the following, for the sake of convenience in calculations in section 3, we decouple multilinear equation (4) into bilinear form by introducing two auxiliary variables x and y:

$$\left(D_x\,\mathrm{e}^{\frac{1}{2}D_m}-\mathrm{e}^{D_n-\frac{1}{2}D_m}+\mathrm{e}^{D_n+\frac{1}{2}D_m}\right)f\cdot f=0, \tag{11}$$

$$\left(D_y\,\mathrm{e}^{\frac{1}{2}D_n}-\mathrm{e}^{D_m-\frac{1}{2}D_n}+\mathrm{e}^{D_m+\frac{1}{2}D_n}\right)f\cdot f=0, \tag{12}$$

where we have assumed that $D_x+D_y=2D_t$. In this case, the Lax pair (9)–(10) is also decoupled to be

$$\psi(m+1,n+1)+\psi(m,n)=\mathrm{e}^{-u(m,n)}[\psi(m+1,n)+\psi(m,n+1)], \tag{13}$$

$$\psi_x(m,n)=\mathrm{e}^{\Delta_n(\phi(m,n)-\phi(m,n-1))}[\psi(m,n-1)-\psi(m,n+1)], \tag{14}$$

$$\psi_y(m,n)=\mathrm{e}^{\Delta_m(\phi(m,n)-\phi(m-1,n))}[\psi(m-1,n)-\psi(m+1,n)]. \tag{15}$$

It should be emphasized that (4) is implied by (11) and (12) but not the other way around.

3. DKP-type pfaffian solution

In this section, we will present a DKP-type pfaffian and then show that it satisfies the bilinear equations (11)–(12). This will be done by calculating the effect of differential and difference operations on the pfaffian and then writing the bilinear equations as quadratic identities for pfaffians. By doing this, we will have established a class of solutions of the (2+1)-dimensional LV equation (3). The reader is referred to [19] for background information on the techniques used in this section.

For the rest of the paper, for brevity of presentation, we will use the subscript m or n to denote an increment in that discrete variable and $\bar{m}$ or $\bar{n}$ to denote an decrement. For example,

$$f_m \equiv f(m+1,n), \qquad f_{\bar{n}} \equiv f(m,n-1) \qquad \text{and} \qquad f_{\bar{m}n} \equiv f(m-1,n+1).$$

Let θ_i, $i = 1, \ldots, 2N$, where N is an arbitrary positive integer, be eigenfunctions of the Lax pair (13)–(15) with potential

$$u = \ln\left(\frac{\tau\tau_{mn}}{\tau_m\tau_n}\right). \tag{16}$$

That is, θ_i satisfy the following equations:

$$\theta_{i,mn} + \theta_i = \frac{\tau_m\tau_n}{\tau\tau_{mn}}(\theta_{i,m} + \theta_{i,n}), \tag{17}$$

$$\theta_{i,x} = \frac{\tau_n\tau_{\bar{n}}}{\tau^2}(\theta_{i,\bar{n}} - \theta_{i,n}), \tag{18}$$

$$\theta_{i,y} = \frac{\tau_m\tau_{\bar{m}}}{\tau^2}(\theta_{i,\bar{m}} - \theta_{i,m}) \tag{19}$$

and τ is any solution of (11)–(12).

We take the pfaffian element (i, j) to be defined by the relations

$$(i,j)_n = (i,j) + \theta_{i,n}\theta_j - \theta_i\theta_{j,n}, \tag{20}$$

$$(i,j)_m = (i,j) - \theta_{i,m}\theta_j + \theta_i\theta_{j,m}, \tag{21}$$

$$(i,j)_x = \frac{\tau_n\tau_{\bar{n}}}{\tau^2}(\theta_{i,\bar{n}}\theta_{j,n} - \theta_{i,n}\theta_{j,\bar{n}}), \tag{22}$$

$$(i,j)_y = \frac{\tau_m\tau_{\bar{m}}}{\tau^2}(\theta_{i,m}\theta_{j,\bar{m}} - \theta_{i,\bar{m}}\theta_{j,m}). \tag{23}$$

It may be shown that these four relations define (i, j) in a consistent way. We can check that all of the compatibility relations $\theta_{i,xy} = \theta_{i,yx}$, $((i,j)_n)_m = ((i,j)_m)_n$, $((i,j)_n)_x = ((i,j)_x)_n$, $((i,j)_n)_y = ((i,j)_y)_n$, $((i,j)_m)_x = ((i,j)_x)_m$, $((i,j)_m)_y = ((i,j)_y)_m$ and $(i,j)_{xy} = (i,j)_{yx}$ are satisfied by detailed calculations.

Consider now the pfaffian of arbitrary order $\sigma = (1, 2, \ldots, 2N)$. We will show that for any τ satisfying equations (11)–(12)

$$f = \sigma\tau = (1, 2, \ldots, 2N)\tau, \tag{24}$$

satisfies the same equations. This means that if u is any solution of the symmetric (2+1)-dimensional LV equation (3) then, using the dependent variable transformation (16),

$$\ln\frac{\sigma\sigma_{mn}}{\sigma_m\sigma_n} + u$$

is also a solution.

In order to prove that f satisfies equations (11)–(12), first we examine the differential and difference formula for σ. We will express these derivatives and differences in terms of the pfaffian index c_i^j defined by $\left(c_i^j, k\right) = \theta_k(m+i, n+j)$ and $\left(c_i^j, c_k^l\right) = 0$.

We may rewrite equations (20)–(23) as

$$(i,j)_n = (i,j) + \left(c_0^0, c_0^1, i, j\right), \qquad (i,j)_m = (i,j) + \left(c_1^0, c_0^0, i, j\right), \tag{25}$$

$$(i,j)_x = \frac{\tau_n\tau_{\bar n}}{\tau^2}\big(c_0^1, c_0^{-1}, i, j\big), \qquad (i,j)_y = \frac{\tau_m\tau_{\bar m}}{\tau^2}\big(c_{-1}^0, c_1^0, i, j\big), \tag{26}$$

and we may also calculate

$$(i,j)_{\bar n} = (i,j) + \big(c_0^0, c_0^{-1}, i, j\big), \qquad (i,j)_{\bar m} = (i,j) + \big(c_{-1}^0, c_0^0, i, j\big), \tag{27}$$

$$(i,j)_{mn} = (i,j) + \frac{\tau_m\tau_n}{\tau\tau_{mn}}\big(c_1^0, c_0^1, i, j\big), \tag{28}$$

$$(i,j)_{m\bar n} = (i,j) + \frac{\tau_m\tau_{\bar n}}{\tau\tau_{m\bar n}}\big(c_1^0, c_0^{-1}, i, j\big), \tag{29}$$

$$(i,j)_{\bar m n} = (i,j) + \frac{\tau_{\bar m}\tau_n}{\tau\tau_{\bar m n}}\big(c_{-1}^0, c_0^1, i, j\big), \tag{30}$$

$$\begin{aligned}(i,j)_{mx} = {}& \frac{\tau_n\tau_{\bar n}}{\tau^2}\big[\big(c_0^1, c_1^0, i, j\big) + \big(c_1^0, c_0^{-1}, i, j\big) + \big(c_0^1, c_0^{-1}, i, j\big)\big] + \frac{\tau_n\tau_{m\bar n}}{\tau\tau_m}\big(c_0^0, c_0^1, i, j\big) \\ & + \frac{\tau_{\bar n}\tau_{m,n}}{\tau\tau_m}\big(c_0^{-1}, c_0^0, i, j\big) + \left(\frac{\tau_n\tau_{m\bar n}}{\tau\tau_m} - \frac{\tau_{\bar n}\tau_{mn}}{\tau\tau_m}\right)\big(c_0^0, c_1^0, i, j\big),\end{aligned} \tag{31}$$

$$\begin{aligned}(i,j)_{ny} = {}& \frac{\tau_m\tau_{\bar m}}{\tau^2}\big[\big(c_{-1}^0, c_1^0, i, j\big) + \big(c_0^1, c_1^0, i, j\big) + \big(c_{-1}^0, c_0^1, i, j\big)\big] + \frac{\tau_m\tau_{\bar m n}}{\tau\tau_n}\big(c_1^0, c_0^0, i, j\big) \\ & + \frac{\tau_{\bar m}\tau_{mn}}{\tau\tau_n}\big(c_0^0, c_{-1}^0, i, j\big) + \left(\frac{\tau_m\tau_{\bar m n}}{\tau\tau_n} - \frac{\tau_{\bar m}\tau_{mn}}{\tau\tau_n}\right)\big(c_0^1, c_0^0, i, j\big).\end{aligned} \tag{32}$$

These results are then used to give expressions for the derivatives and differences of the pfaffian σ. We will write $\sigma = (1, 2, \ldots, 2N) \equiv (\cdot)$ for short and extend this to write $\big(c_0^1, c_1^0, 1, 2, \ldots, 2N\big) \equiv \big(c_0^1, c_1^0, \cdot\big)$ and so on. Using equations (25)–(32), we obtain

$$\sigma_n = (\cdot) + \big(c_0^0, c_0^1, \cdot\big), \qquad \sigma_m = (\cdot) + \big(c_1^0, c_0^0, \cdot\big), \tag{33}$$

$$\sigma_x = \frac{\tau_n\tau_{\bar n}}{\tau^2}\big(c_0^1, c_0^{-1}, \cdot\big), \qquad \sigma_y = \frac{\tau_m\tau_{\bar m}}{\tau^2}\big(c_{-1}^0, c_1^0, \cdot\big), \tag{34}$$

$$\sigma_{\bar n} = (\cdot) + \big(c_0^0, c_0^{-1}, \cdot\big), \qquad \sigma_{\bar m} = (\cdot) + \big(c_{-1}^0, c_0^0, \cdot\big), \tag{35}$$

$$\sigma_{mn} = (\cdot) + \frac{\tau_m\tau_n}{\tau\tau_{mn}}\big(c_1^0, c_0^1, \cdot\big), \tag{36}$$

$$\sigma_{m\bar n} = (\cdot) + \frac{\tau_m\tau_{\bar n}}{\tau\tau_{m\bar n}}\big(c_1^0, c_0^{-1}, \cdot\big), \tag{37}$$

$$\sigma_{\bar m n} = (\cdot) + \frac{\tau_{\bar m}\tau_n}{\tau\tau_{\bar m n}}\big(c_{-1}^0, c_0^1, \cdot\big), \tag{38}$$

$$\begin{aligned}\sigma_{mx} = {}& \frac{\tau_n\tau_{\bar n}}{\tau^2}\big[\big(c_0^1, c_1^0, \cdot\big) + \big(c_1^0, c_0^{-1}, \cdot\big) + \big(c_0^1, c_0^{-1}, \cdot\big)\big] + \frac{\tau_n\tau_{m\bar n}}{\tau\tau_m}\big(c_0^0, c_0^1, \cdot\big) + \frac{\tau_{\bar n}\tau_{mn}}{\tau\tau_m}\big(c_0^{-1}, c_0^0, \cdot\big) \\ & + \left(\frac{\tau_n\tau_{m\bar n}}{\tau\tau_m} - \frac{\tau_{\bar n}\tau_{mn}}{\tau\tau_m}\right)\big(c_0^0, c_1^0, \cdot\big) + \frac{\tau_n\tau_{\bar n}}{\tau^2}\big(c_1^0, c_0^0, c_0^1, c_0^{-1}, \cdot\big),\end{aligned} \tag{39}$$

$$\begin{aligned}\sigma_{ny} = {}& \frac{\tau_m\tau_{\bar m}}{\tau^2}\big[\big(c_{-1}^0, c_1^0, \cdot\big) + \big(c_0^1, c_1^0, \cdot\big) + \big(c_{-1}^0, c_0^1, \cdot\big)\big] + \frac{\tau_m\tau_{\bar m n}}{\tau\tau_n}\big(c_1^0, c_0^0, \cdot\big) + \frac{\tau_{\bar m}\tau_{mn}}{\tau\tau_n}\big(c_0^0, c_{-1}^0, \cdot\big) \\ & + \left(\frac{\tau_m\tau_{\bar m n}}{\tau\tau_n} - \frac{\tau_{\bar m}\tau_{mn}}{\tau\tau_n}\right)\big(c_0^1, c_0^0, \cdot\big) + \frac{\tau_m\tau_{\bar m}}{\tau^2}\big(c_0^0, c_0^1, c_{-1}^0, c_1^0, \cdot\big).\end{aligned} \tag{40}$$

Using equations (33)–(40) and the fact that τ itself is a solution of equations (11)–(12), substituting equation (24) into equations (11)–(12) leads to the following two pfaffian identities:

$$\begin{aligned}&\big(c_1^0, c_0^0, c_0^1, c_0^{-1}, 1, 2, \ldots, 2N\big)(1, 2, \ldots, 2N) - \big(c_1^0, c_0^0, 1, 2, \ldots, 2N\big)\big(c_0^1, c_0^{-1}, 1, 2, \ldots, 2N\big)\\&\quad + \big(c_1^0, c_0^1, 1, 2, \ldots, 2N\big)\big(c_0^0, c_0^{-1}, 1, 2, \ldots, 2N\big) - \big(c_1^0, c_0^{-1}, 1, 2, \ldots, 2N\big)\\&\quad \times \big(c_0^0, c_0^1, 1, 2, \ldots, 2N\big) = 0,\\&\big(c_0^0, c_0^1, c_{-1}^0, c_1^0, 1, 2, \ldots, 2N\big)(1, 2, \ldots, 2N) - \big(c_0^0, c_0^1, 1, 2, \ldots, 2N\big)\big(c_{-1}^0, c_1^0, 1, 2, \ldots, 2N\big)\\&\quad + \big(c_0^0, c_{-1}^0, 1, 2, \ldots, 2N\big)\big(c_0^1, c_1^0, 1, 2, \ldots, 2N\big) - \big(c_0^0, c_1^0, 1, 2, \ldots, 2N\big)\\&\quad \times \big(c_0^1, c_{-1}^0, 1, 2, \ldots, 2N\big) = 0.\end{aligned}$$

Thus we have proved that $f = (1, 2, \ldots, 2N)\tau$ actually satisfies the bilinear symmetric LV system.

Here we would like to point out that the pfaffian solution given here is more general than the solutions we would find if we followed the approach taken by Ohta [12] for the NVN equation in the continuous case. These solutions correspond to the special case $\tau = 1$ in our approach. The approach we take is closer to the approach of Athorne and Nimmo [11]. This special case is considered in the next section.

4. Explicit solutions

In the last section we proved that solutions of the (2+1)-dimensional LV equation (3) are given by the formula

$$u = \ln \frac{f_{mn} f}{f_m f_n}, \tag{41}$$

where $f = (1, 2, \ldots, 2N)\tau$ in which τ is any solution of equations (11) and (12) and $(1, 2, \ldots, 2N)$ is the pfaffian with element defined by (20)–(23). Observe that u given by (41) is invariant under the symmetries $f \to a(n, x) f$ and $f \to b(m, y) f$. The solutions we find are similar in some respects to those obtained in the continuous case that are described in [11, 12]. These two papers take a slightly different approach to the solutions. We follow the approach in [11].

In this paper, the only explicit solutions we will discuss arise when the vacuum solution is $u = \ln(\tau_{mn}\tau/\tau_m\tau_n) = 0$, given by choosing $\tau = 1$ also. In this case, (17) may be written as

$$\Delta_m \Delta_n \theta_i = 0,$$

which implies that each θ_i is the sum of functions only depending on n and m respectively. By considering (17) and (18) also, we have that

$$\theta_i(m, n, x, y) = \phi_i(n, x) + \psi_i(m, y),$$

where each ϕ_i and ψ_i satisfy

$$\phi_x = \phi_{\bar{n}} - \phi_n, \tag{42}$$

$$\psi_y = \psi_{\bar{m}} - \psi_m, \tag{43}$$

respectively. The simplest choices of these solutions are

$$\phi_i = \alpha_i \left(\frac{1 - p_i}{1 + p_i}\right)^{-n} \exp\left(\frac{-4 p_i x}{1 - p_i^2}\right), \tag{44}$$

$$\psi_i = \beta_i \left(\frac{1 - q_i}{1 + q_i}\right)^{-m} \exp\left(\frac{-4 q_i y}{1 - q_i^2}\right), \tag{45}$$

in which α_i, β_i, p_i and q_i are constants, and we take eigenfunctions to be $\theta_i = \phi_i + \psi_i$.

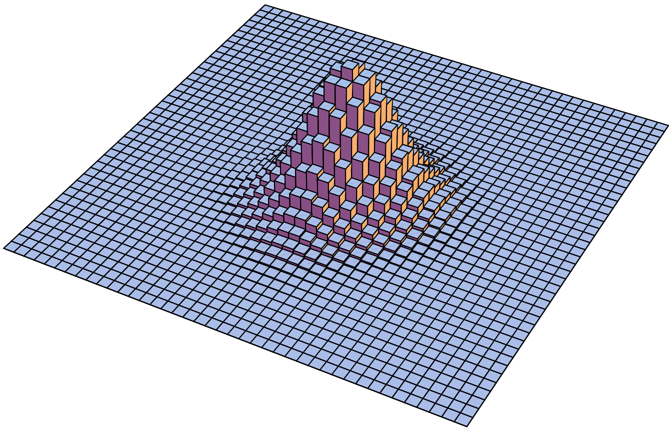

Figure 1. A discrete dromion.

Remark. The above parametrization of the dispersion relation will be used rather than the more obvious one,

$$\phi = \alpha k^n \exp((k^{-1} - k)x), \qquad \psi = \beta \ell^m \exp((\ell^{-1} - \ell)y).$$

Its form is given by the Miwa transformation [23] relating continuous and discrete dependence in BKP/DKP hierarchies. This is not essential to the calculations performed here, but it gives a more familiar look to the form of the terms in the pfaffian element.

Although in the previous section we were only able to define the pfaffian element (i, j) implicitly through the relations (20)–(23), for these eigenfunctions we may calculate it explicitly. From these relations we get

$$(i, j) = \frac{p_i - p_j}{p_i + p_j}\phi_i\phi_j - \frac{q_i - q_j}{q_i + q_j}\psi_i\psi_j + \phi_i\psi_j - \psi_i\phi_j + C_{i,j},$$

where $C_{i,j}$ is an arbitrary constant and $C_{ij} = -C_{ji}$. The pfaffian $f = (1, 2, \ldots, 2N)$ with this pfaffian element defines a class of solution including both solitons and dromions. For example, consider the case $C_{12} = 0$, then

$$(1, 2) = \frac{p_1 - p_2}{p_1 + p_2}\phi_1\phi_2 - \frac{q_1 - q_2}{q_1 + q_2}\psi_1\psi_2 + \phi_1\psi_2 - \psi_1\phi_2.$$

Through the transformation (41), and appropriate choice for the signs of α_i, β_j, this gives the same solution u as

$$f = 1 + \frac{\phi_1}{\psi_1} + \frac{\phi_2}{\psi_2} + \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 + p_2)(q_1 + q_2)}\frac{\phi_1}{\psi_1}\frac{\phi_2}{\psi_2}.$$

This is a two-soliton solution. Note also that when we set $p_2 = q_2 = 0$, we obtain (up to an irrelevant factor)

$$f = 1 + \frac{\phi_1}{\psi_1},$$

which is a one-soliton solution. In a similar way we obtain the N-soliton solution for any N.

The (1, 1)-dromion solution is obtained from $f = (1, 2)$ with the choice $q_1 = p_2 = 0$. This is similar to, but more general than, the two-soliton solution consisting of one soliton propagating in each of the m- and n-directions. We have

$$(1, 2) = \phi_1 + \psi_2 + \phi_1\psi_2 + C_{1,2},$$

or equivalently,

$$f = 1 + \phi_1 + \psi_2 + A\phi_1\psi_2,$$

where A is an arbitrary constant. A plot of a discrete dromion is shown in figure 1.

The general (M, N)-dromion when $M + N$ is even is obtained by taking the pfaffian $(1, 2, \ldots, M + N)$ with $q_1 = \cdots = q_M = p_{M+1} = \cdots = p_{M+N} = 0$. As for the soliton solutions, we obtain the solution in the case $M + N$ odd by setting one more p_i or q_i equal to zero.

5. Conclusion

We have proposed a symmetric LV equation with two discrete variables and one continuous variable. The equation may be viewed as a strong (2+1)-dimensional generalization of the Lotka–Volterra equation in the sense that two discrete variables appear on an equal footing. It has been shown that the (2+1)-dimensional LV equation is integrable in the sense of having a Bäcklund transformation and Lax pair. Pfaffian solutions to the (2+1)-dimensional LV equation have been derived. As a special reduction of the pfaffian solutions we obtain discrete soliton and discrete dromions solutions.

Also, we can show that the continuous analogue of the (2+1)-dimensional LV equation is the NVN equation (1). This can be seen most easily from the bilinear form by setting $D_m = \delta D_\eta$, $D_n = \epsilon D_\xi$, $D_x = -\frac{1}{3}\epsilon^3 D_{x_3} - 2\epsilon D_\xi$, $D_y = -\frac{1}{3}\delta^3 D_{y_3} - 2\delta D_\eta$ in (11) and (12) to obtain

$$\frac{1}{\delta}\sinh\left(\frac{1}{2}\delta D_\eta\right)\frac{1}{\epsilon^3}\left[-\frac{1}{3}\epsilon^3 D_{x_3} - 2\epsilon D_\xi + 2\sinh(\epsilon D_\xi)\right] f \cdot f = 0, \tag{46}$$

$$\frac{1}{\epsilon}\sinh\left(\frac{1}{2}\epsilon D_\xi\right)\frac{1}{\delta^3}\left[-\frac{1}{3}\delta^3 D_{y_3} - 2\delta D_\eta + 2\sinh(\delta D_\eta)\right] f \cdot f = 0. \tag{47}$$

In the limit $\epsilon \to 0$, $\delta \to 0$, the leading order behaviour in (46) and (47) gives

$$D_\eta\big(D_{x_3} - D_\xi^3\big) f \cdot f = 0, \tag{48}$$

$$D_\xi\big(D_{y_3} - D_\eta^3\big) f \cdot f = 0, \tag{49}$$

which is nothing but bilinear form of the NVN equation [12].

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Appendix. Hirota's bilinear operator identities

The following bilinear operator identities hold for arbitrary functions a, b, c and d.

$$\big[\sinh\big(\tfrac{1}{2}D_n\big)a \cdot b\big]\big[e^{\frac{1}{2}D_n}c \cdot c\big] - \big[\sinh\big(\tfrac{1}{2}D_n\big)d \cdot c\big]\big[e^{\frac{1}{2}D_n}b \cdot b\big] = \sinh\big(\tfrac{1}{2}D_n\big)(ac - db) \cdot bc \tag{A.1}$$

$$\mathrm{e}^{\frac{1}{2}D_n}\big(\mathrm{e}^{\frac{1}{2}D_m}a\cdot a\big)\cdot\big(\mathrm{e}^{\frac{1}{2}D_m}a\cdot a\big)=\mathrm{e}^{\frac{1}{2}D_m}\big(\mathrm{e}^{\frac{1}{2}D_n}a\cdot a\big)\cdot\big(\mathrm{e}^{\frac{1}{2}D_n}a\cdot a\big) \tag{A.2}$$

$$\big[\sinh\big(\tfrac{1}{2}D_m\big)a\cdot b\big]\big[\mathrm{e}^{\frac{1}{2}D_m}c\cdot c\big]-\big[\sinh\big(\tfrac{1}{2}D_m\big)d\cdot c\big]\big[\mathrm{e}^{\frac{1}{2}D_m}b\cdot b\big]=\sinh\big(\tfrac{1}{2}D_m\big)(ac-db)\cdot bc \tag{A.3}$$

$$\sinh\big(\tfrac{1}{2}D_m\big)\big[D_t\,\mathrm{e}^{\frac{1}{2}D_n}a\cdot a\big]\cdot\big[\mathrm{e}^{\frac{1}{2}D_n}a\cdot a\big]=\sinh\big(\tfrac{1}{2}D_n\big)\big[D_t\,\mathrm{e}^{\frac{1}{2}D_m}a\cdot a\big]\cdot\big[\mathrm{e}^{\frac{1}{2}D_m}a\cdot a\big] \tag{A.4}$$

$$\big[D_t\,\mathrm{e}^{\frac{1}{2}D_m}a\cdot a\big]\big[\mathrm{e}^{\frac{1}{2}D_m}b\cdot b\big]-\big[D_t\,\mathrm{e}^{\frac{1}{2}D_m}b\cdot b\big]\big[\mathrm{e}^{\frac{1}{2}D_m}a\cdot a\big]=2\sinh\big(\tfrac{1}{2}D_m\big)(D_ta\cdot b)\cdot ab \tag{A.5}$$

$$\begin{aligned}&\big[\mathrm{e}^{D_n-\frac{1}{2}D_m}a\cdot a\big]\big[\mathrm{e}^{\frac{1}{2}D_m}b\cdot b\big]-\big[\mathrm{e}^{D_n-\frac{1}{2}D_m}b\cdot b\big]\big[\mathrm{e}^{\frac{1}{2}D_m}a\cdot a\big]\\&\qquad=2\sinh\big(\tfrac{1}{2}D_n\big)\big[\mathrm{e}^{\frac{1}{2}D_n-\frac{1}{2}D_m}a\cdot b\big]\cdot\big[\mathrm{e}^{-\frac{1}{2}D_n+\frac{1}{2}D_m}a\cdot b\big]\end{aligned} \tag{A.6}$$

$$\begin{aligned}&\big[\mathrm{e}^{D_n+\frac{1}{2}D_m}a\cdot a\big]\big[\mathrm{e}^{\frac{1}{2}D_m}b\cdot b\big]-\big[\mathrm{e}^{D_n+\frac{1}{2}D_m}b\cdot b\big]\big[\mathrm{e}^{\frac{1}{2}D_m}a\cdot a\big]\\&\qquad=2\sinh\big(\tfrac{1}{2}D_n\big)\big[\mathrm{e}^{\frac{1}{2}D_n+\frac{1}{2}D_m}a\cdot b\big]\cdot\big[\mathrm{e}^{-\frac{1}{2}D_n-\frac{1}{2}D_m}a\cdot b\big]\end{aligned} \tag{A.7}$$

$$\big[\mathrm{e}^{\frac{1}{2}D_m}a\cdot a\big]\big[\mathrm{e}^{\frac{1}{2}D_m}b\cdot b\big]=\big[\mathrm{e}^{\frac{1}{2}D_m}ab\cdot ab\big] \tag{A.8}$$

$$\sinh\big(\tfrac{1}{2}D_m\big)\big[\sinh\big(\tfrac{1}{2}D_n\big)a\cdot b\big]\cdot\big[\mathrm{e}^{\frac{1}{2}D_n}b\cdot b\big]=\sinh\big(\tfrac{1}{2}D_n\big)\big[\sinh\big(\tfrac{1}{2}D_m\big)a\cdot b\big]\cdot\big[\mathrm{e}^{\frac{1}{2}D_m}b\cdot b\big] \tag{A.9}$$

$$\sinh\big(\tfrac{1}{2}D_m\big)a\cdot a=0. \tag{A.10}$$

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